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Pressure Distribution and Flow Development in Unsteady Incompressible Laminar Boundary Layers

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It is shown that in an unsteady flow the friction drag is always accompanied by the form drag whose magnitude is comparable to that of the former and that the pressure around the unsteady boundary layer can be far from that of the inviscid irrotational flow. The unsteady boundary-layer equations and boundary conditions for the external potential flow are modified accordingly and the flow around a circular cylinder which is set impulsively to move in a constant velocity is analysed using these modified boundary-layer equations. The solutions are in power series of $\sqrt{\tau}$ rather than τ where τ is the dimensionless time elapsed since the onset of motion, and the form drag, like the friction drag, decreases from infinity in inverse proportion to $\sqrt{\tau}$, when τ is small.

Key Words: Starting Flow, Circular Cylinder, Pressure Distribution, Form Drag, Boundary-Layer Formation

1. Introduction

In the boundary-layer theory, it is generally accepted that the pressure gradient $\partial p/\partial x$ along the streamwise direction can be computed from the approximate equation

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial x}, \quad (1)$$

where the x-component velocity u_e just outside the boundary layer is regarded as given (Batchelor, 1967, p. 305; White, 1991, p. 230). In other words, the pressure is thought to be determinable without considering the detail of the boundary-layer flow. Thus Blasius(1908) and many later investigators analysed the initial flow around, say, a circular cylinder starting impulsively from rest to move in a constant velocity, assuming that the pressure in and around the boundary layer does not vary with time, i.e.,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=u_e\frac{\partial u_e}{\partial x}$$

The form drag is also thought to be zero just after the start of motion but increase with time as the boundary layer thickens and the flow separates (Stuart, 1963, Sec. VII. 7).

On the other hand, Collins and Dennis(1973) solved the full Navier-Stokes equations for the initial flow over an impulsively started circular cylinder using boundary-layer variables and showed that both the friction and pressure drag are infinite at the start of motion, implying that pressure distribution around the cylinder will be different from that of the inviscid irrotational flow. Bar-Lev and Yang(1975) solved the same problem by the method of matched asymptotic expansions and got similar results on the form drag. Smith and Stansby(1988) simulated the same flow by a Lagrangian vortex method and got drag coefficients which are in agreement with analytic solutions at small times. However, neither of them gave any physical reasons for the appearance of the form drag nor the explanation

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about the pressure field around the cylinder : they got results simply by integrating the vorticity gradient around the cylinder.

In the present paper, we will show that, in unsteady incompressible flow, the friction drag should be always accompanied by the form drag whose magnitude is comparable to that of the former. Another purpose of the present paper is to modify the unsteady boundary-layer equations accordingly so that the pressure variation can be accounted for : the external velocity u_e is modified as

$$u_e = u_{ep} + u_{ef}, \tag{2}$$

where u_{ep} is the external velocity resulting from a first-order potential flow calculation and u_{ef} denotes the correction due to the boundary layer. This modification itself is not new. It is adopted in the modern interactive boundary-layer theory (Stewartson, 1974 ; Veldman, 1981 ; Henkes and Veldman, 1987). However, its applications are mostly to steady flows, where the effect of u_{ef} on the pressure field around the boundary layer is minor and localized to the small region of the extent $O(Re^{-3/8}L)$ (Re : the Reynolds number, L : the characteristic length of the flow) around singular points such as trailing edges and points of separation (Stewartson, 1974). In an unsteady flow, on the other hand, $\partial u_{ef}/\partial t$, on which the pressure gradient $\partial p/\partial x$ depends also, can be large, even though u_{ef} itself is small : in an unsteady boundary layer around an impulsively started body, $\partial u_{ef}/\partial t$ is $O(\partial\delta/\partial t) = O(\nu^{1/2}t^{-1/2})$ and thus infinite just after the start of motion, where u_{ef} is only of the order of the boundary-layer thickness δ.

In the next section, it is shown, using the momentum theorem, that in an unsteady incompressible flow the friction drag should be always accompanied by the form drag whose magnitude is comparable to that of the former and that the pressure field around the unsteady incompressible boundary layer can be far from that of the inviscid irrotational flow. In Sec. 3, the unsteady boundary-layer equations are modified accordingly and a procedure to solve these modified equations is suggested. In Sec. 4, these modified equations are used to analyse the unsteady flow around a circular cylinder starting impulsively from rest to a uniform motion and conclusions are given in Sec. 5.

2. Balance of Momentum in an Unsteady Flow Around a Body

Suppose that a body in an unbounded incompressible viscous fluid with density ρ and kinematic viscosity ν has started to move rectilinearly sometime ago and is now in motion with velocity $-U(t)\boldsymbol{k}$, where t is a time elapsed since the start of motion. The body, then, will experience a drag $D\boldsymbol{k}$ and we want to know how this drag is transferred into the fluid. To this end the momentum theorem can be used. As a system of reference we choose the coordinates fixed in space so that the velocity at infinity is zero and for the control volume V the region between a cylinder with the curved surface A_S parallel to and plane faces of area A_F normal to $\boldsymbol{k}$ and the closed surface S which is coincident with the body surface instantly (Fig. 1). The momentum theorem then gives, neglecting small viscous forces acting at the cylindrical surface A_S,

$$D = -\frac{\partial}{\partial t}\int_V \rho w dV + \int_S \rho w \boldsymbol{v}\cdot\boldsymbol{n} dS + \int_{A_F} (p_1 + \rho w_1^2 - p_2 - \rho w_2^2)\, dA_F - \int_{A_S} \rho w \boldsymbol{v}\cdot\boldsymbol{n} dA_S, \tag{3}$$

where w_1, p_1, and w_2, p_2 are the values of the $\boldsymbol{k}$-component of the velocity $\boldsymbol{v}$ and the pressure

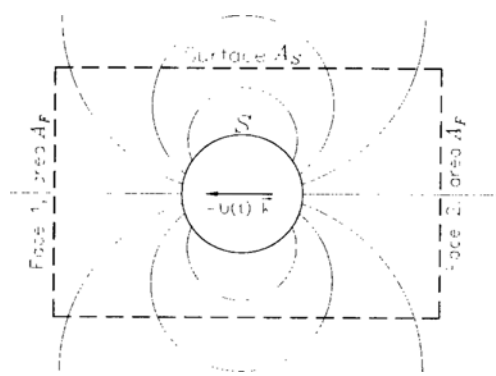

Fig. 1 Control surface (broken line) to get the drag on a body moving through a fluid

at the upstream and downstream faces respectively. On the surface S the unit normal $\boldsymbol{n}$ is into the control volume (outward from the body).

As the motion started not long before, the vorticity generated at the solid surface has not been transferred far, but is clustered near the body and the wake is short, too. Thus we can take the cylinder big enough to enclose completely the region where the flow is rotational. Since there are no sources and sinks, the irrotational flow at large distances r from the body will be like that of a source doublet and falls off as r^{-3} in three dimensions and r^{-2} in two dimensions. Thus the total momentum flux through the surface and faces of the cylinder tends to zero as the areas A_S and A_F go to infinity. On the body surface the fluid velocity is equal to $-U\boldsymbol{k}$, the moving velocity of the body, and the second integral of Eq. (3) becomes zero. Thus Eq. (3) becomes

$$D=-\frac{\partial}{\partial t}\int_V \rho w dV+\int_{A_F}(p_1-p_2)\,dA_F, \quad (4)$$

as the control volume goes to infinity. Furthermore, if we let the length of the cylinder go to infinity first and then the area A_F of the face, the pressure integral goes to zero and we have

$$D=-\frac{\partial}{\partial t}\left(\lim_{V\to\infty}\int_V \rho w dV\right) \quad (5)$$

Thus in an unsteady flow the drag is balanced by the increase of the momentum. Suppose now that the wake has not been formed yet and the displacement thickness δ_1 is relatively small. Then the total momentum of the fluid will not change, if we replace the real flow around the body by the uniform motion with velocity $-U\boldsymbol{k}$ of the mass of fluid between S and the surface δ_1 distant from S and the potential flow around the virtual body formed by the body and this uniformaly moving mass of fluid. Thus Eq. (5) can be written as

$$D=\frac{\partial}{\partial t}\left(U\int_S \rho\delta_1 dS\right)-\frac{\partial}{\partial t}\left(\lim_{V'\to\infty}\int \rho w' dV'\right)$$
$$=D_f+D_p, \quad (6)$$

where the second integral is over the volume V' outside the virtual body and w' is the $\boldsymbol{k}$-component velocity of the potential flow around it. The second integral of Eq. (6) gives the added mass of the virtual body multiplied by the velocity U (Milne-Thomson, 1968, p. 491) and, hence, D_p denotes the increasing rate of the momentum due to growth of the added mass as well as the acceleration of the body. Considering that the pressure is approximately uniform across the boundary layer and that the reaction in the potential flow is the resultant of pressure forces around the body, we can easily anticipate that D_p is equal to the form drag and, hence, D_f the friction drag. Thus, since the added mass is propotional to the volume of the body (in this case, of the virtual body) for the given shape (Batchelor, 1967, Sec. 6. 4), the form drag D_p is not zero but of the order of the friction drag D_f, even if the body moves in a constant velocity.

The fact that the actual pressure field will be far from that of the inviscid irrotational flow can be inferred again as follows. Suppose now that the body is placed in the cylindrical tube A_S with cross-sectional area A_F and the fluid comes with uniform velocity $U\boldsymbol{k}$ from the left. Then, neglecting viscous forces acting at the cylindrical surface A_S again, we have

$$D=-\frac{\partial}{\partial t}\int_V \rho w dV$$
$$+\int_{A_F}(p_1+\rho w_1^2-p_2-\rho w_2^2)\,dA_F.$$

But, in this case, the continuity equation gives

$$-\frac{\partial}{\partial t}\int_V \rho w dV=\frac{\partial(\rho UV)}{\partial t}$$

and we have, for constant U,

$$D=\int_{A_F}(p_1+\rho w_1^2-p_2-\rho w_2^2)\,dA_F$$

Schwarz's inequality (Jeffreys and Jefferys, 1978, p. 54), however, gives

$$\int_{A_F}(\rho w_1^2-\rho w_2^2)\,dA_F$$
$$=\int_{A_F}\rho(U^2-w_2^2)\,dA_F\leq 0$$

and we have

$$D\leq\int_{A_F}(p_1-p_2)\,dA_F,$$

or, letting the wall of the tube to recede,

$$D\leq\lim_{A_F\to\infty}\int_{A_F}(p_1-p_2)\,dA_F \quad (7)$$

Thus the mean pressure upstream of the body moving in a constant velocity should always be higher than that downstream of the body.

3. Unsteady Boundary-Layer Equations

Suppose that a solid body immersed in an unbounded fluid is made to move rectilinearly with velocity $-U(t)\boldsymbol{k}$. Let the surface of the body be denoted by S. Then the position of a point P near the surface can be described by curvilimear coordinates (x, y, z), where x and z are orthogonal curvilinear coordinates on S and y is measured along the unit normal $\boldsymbol{n}$ to S. Scale factors and unit vectors tangential to $x-$ and $z-$coordinate lines are denoted by h_x, h_y and $\boldsymbol{a}$, $\boldsymbol{b}$ respectively with $\boldsymbol{a}\times\boldsymbol{n}=\boldsymbol{b}$ and velocity components u, v, and w (in $x-$, $y-$, and $z-$direction, respectively) are relative to the body.

Then the boundary-layer equations and continuity equation become (Crabtree et al., 1963)

$$\frac{\partial u}{\partial t}+\frac{u}{h_x}\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{w}{h_z}\frac{\partial u}{\partial z}+\frac{uw}{h_xh_z}\frac{\partial h_x}{\partial z}-\frac{w^2}{h_xh_z}\frac{\partial h_z}{\partial x}-\nu\frac{\partial^2 u}{\partial y^2}=\frac{\partial u_e}{\partial t}+\frac{u_e}{h_x}\frac{\partial u_e}{\partial x}+\frac{w_e}{h_z}\frac{\partial u_e}{\partial z}+\frac{u_ew_e}{h_xh_z}\frac{\partial h_x}{\partial z}-\frac{w_e^2}{h_xh_z}\frac{\partial h_z}{\partial x}, \tag{8}$$

$$\frac{\partial w}{\partial t}+\frac{u}{h_x}\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{w}{h_z}\frac{\partial w}{\partial z}-\frac{u^2}{h_xh_z}\frac{\partial h_x}{\partial z}+\frac{uw}{h_xh_z}\frac{\partial h_z}{\partial x}-\nu\frac{\partial^2 w}{\partial y^2}=\frac{\partial w_e}{\partial t}+\frac{u_e}{h_x}\frac{\partial w_e}{\partial x}+\frac{w_e}{h_z}\frac{\partial w_e}{\partial z}-\frac{u_e^2}{h_xh_z}\frac{\partial h_x}{\partial z}+\frac{u_ew_e}{h_xh_z}\frac{\partial h_z}{\partial x}, \tag{9}$$

and

$$\frac{1}{h_xh_z}\left\{\frac{\partial}{\partial x}(h_zu)+\frac{\partial}{\partial z}(h_xw)\right\}+\frac{\partial v}{\partial y}=0, \tag{10}$$

where ν is a kinematic viscosity and u_e and w_e are components of the mainstream velocity just outside the layer parallel to $\boldsymbol{a}$ and $\boldsymbol{b}$ on S respectively. These three equations together with boundary conditions,[1)]

$$u\to u_e,\ w\to w_e,\ \text{as } y\to\infty, \tag{11}$$

and

$$u=0,\ v=0,\ w=0,\ \text{when } y=0, \tag{12}$$

serve to determine u, v, and w in the boundary layer, when external stream velocities u_e and w_e are known.

External stream velocities u_e and w_e can be got from the velocity potential ϕ, as usual. In other words,

$$u_e=-\frac{1}{h_x}\frac{\partial\phi}{\partial x}+U\boldsymbol{k}\cdot\boldsymbol{a}, \tag{13}$$

and

$$w_e=-\frac{1}{h_z}\frac{\partial\phi}{\partial z}+U\boldsymbol{k}\cdot\boldsymbol{b}, \tag{14}$$

where ϕ satisfies the Laplace equation

$$\nabla^2\phi=0 \tag{15}$$

The boundary condition for ϕ at infinity is not changed, either, i.e.,

$$-\nabla\phi|_{r\to\infty}=0, \tag{16}$$

where r is a distance from a material point of the body, But the boundary condition at $y=0$ should be modified : the boundary layer induces the transpiration (or displacement) velocity (Lighthill, 1958)

$$v_{ef}=\lim_{y\to\infty}\left[v+\frac{y}{h_xh_z}\left\{\frac{\partial}{\partial x}(h_zu_e)+\frac{\partial}{\partial z}(h_xw_e)\right\}\right]; \tag{17}$$

just outside the layer and the boundary condition for ϕ at $y=0$ should be

$$-\frac{\partial\phi}{\partial y}\bigg|_{y=0}=-U\boldsymbol{k}\cdot\boldsymbol{n}+v_{ef} \tag{18}$$

Thus the velocity potential ϕ can be put as

$$\phi=\phi_p+\phi_f, \tag{19}$$

where both ϕ_p and ϕ_f satisfy the Laplace equations,

$$\nabla^2\phi_p=0, \tag{20a}$$
$$\nabla^2\phi_f=0, \tag{20b}$$

and boundary conditions,

1) To be strict, the boundary-layer thickness δ is assumed to be small compared with the typical length L of the body so that it is possible to find a range of distance y such that $y/\delta\gg1$ and $y/L\ll1$ simultaneously. Conditions $y\to\infty$ in (11) and (17) and $y=0$ in (18) should be understood as that y falls in this range.

$$-\nabla\phi_p|_{r\to\infty}=0,\quad -\nabla\phi_f|_{r\to\infty}=0, \tag{21}$$

and

$$-\frac{\partial\phi_p}{\partial y}\bigg|_{y=0}=-U\boldsymbol{k}\cdot\boldsymbol{n},\quad -\frac{\partial\phi_f}{\partial y}\bigg|_{y=0}=v_{ef}, \tag{22}$$

respectively.

u_e and w_e can also be put as

$$u_e=u_{ep}+u_{ef},\quad w_e=w_{ep}+w_{ef}, \tag{23}$$

where

$$u_{ep}=-\frac{1}{h_x}\frac{\partial\phi_p}{\partial x}+U\boldsymbol{k}\cdot\boldsymbol{a},$$

$$u_{ef}=\frac{1}{h_x}\frac{\partial\phi_f}{\partial x}, \tag{24b}$$

and

$$w_{ep}=-\frac{1}{h_z}\frac{\partial\phi_p}{\partial z}+U\boldsymbol{k}\cdot\boldsymbol{b}, \tag{25a}$$

$$w_{ef}=\frac{1}{h_x}\frac{\partial\phi_f}{\partial z}, \tag{25b}$$

respectively.

u_{ef}, v_{ef}, and w_{ef} are all of order δ, the boundary-layer thickness, and can be neglected except their time-derivative. Thus Eqs. (8) and (9) become

$$\frac{\partial u}{\partial t}+\frac{u}{h_x}\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{w}{h_z}\frac{\partial u}{\partial z}+\frac{uw}{h_xh_z}\frac{\partial h_x}{\partial z}-\frac{w^2}{h_xh_z}\frac{\partial h_z}{\partial x}-\nu\frac{\partial^2 u}{\partial y^2}$$
$$=\frac{\partial u_{ef}}{\partial t}+\frac{\partial u_{ep}}{\partial t}+\frac{u_{ep}}{h_x}\frac{\partial u_{ep}}{\partial x}+\frac{w_{ep}}{h_z}\frac{\partial u_{ep}}{\partial z}+\frac{u_{ep}w_{ep}}{h_xh_z}\frac{\partial h_x}{\partial z}-\frac{w_{ep}^2}{h_xh_z}\frac{\partial h_z}{\partial x}, \tag{26}$$

$$\frac{\partial w}{\partial t}+\frac{u}{h_x}\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{w}{h_z}\frac{\partial w}{\partial z}-\frac{u^2}{h_xh_z}\frac{\partial h_x}{\partial z}+\frac{uw}{h_xh_z}\frac{\partial h_z}{\partial x}-\nu\frac{\partial^2 w}{\partial y^2}$$
$$=\frac{\partial w_{ef}}{\partial t}+\frac{\partial w_{ep}}{\partial t}+\frac{u_{ep}}{h_x}\frac{\partial w_e}{\partial x}+\frac{w_{ep}}{h_z}\frac{\partial w_{ep}}{\partial z}-\frac{u_{ep}^2}{h_xh_z}\frac{\partial h_x}{\partial z}+\frac{u_{ep}w_{ep}}{h_xh_z}\frac{\partial h_z}{\partial x}, \tag{27}$$

respectively. The momentum equations, Eqs. (26) and (27), and continuity equation, Eq. (10), together with boundary conditions,[2)]

$$u\to u_{ep}+u_{ef},\quad w\to w_{ep}+w_{ef},\ \text{as}\ y\to\infty, \tag{28}$$

and

$$u=0,\quad v=0,\quad w=0,\ \text{when}\ y=0, \tag{29}$$

serve to determine u, v, and w in the boundary layer, where u_{ep}, w_{ep}, u_{ef} and w_{ef} can be got by solving the Laplace equations, Eqs. (20a,b), with boundary conditions (21) and (22). Also the normal velocity v_{ef} (Eq. (17)) becomes

$$v_{ef}=\lim_{y\to\infty}\left[v+\frac{y}{h_xh_z}\left\{\frac{\partial}{\partial x}(h_zu_{ef})+\frac{\partial}{\partial z}(h_xw_{ef})\right\}+\frac{y}{h_xh_z}\left\{\frac{\partial}{\partial x}(h_zu_{ep})+\frac{\partial}{\partial z}(h_xw_{ep})\right\}\right] \tag{30}$$

Thus the external and boundary-layer flow are coupled through u_{ep}, w_{ep}, u_{ef}, w_{ef} and v_{ef}: u_{ef} and w_{ef} are got by solving the Laplace equation, Eq. (20b), and used together with u_{ep} and w_{ep} as boundary conditions for the boundary-layer flow, while the boundary-layer flow gives v_{ef}, the boundary condition for the external potential flow. Finally, the pressure p can be computed from the Bernoulli equation for the unsteady irrotational flow

$$\frac{p-p_\infty}{\rho}=\frac{\partial\phi_f}{\partial t}+\frac{\partial\phi_p}{\partial t}-\frac{1}{2}(\nabla\phi_p)^2, \tag{31}$$

where p_∞ is the pressure at infinity.

Neglect of terms having any powers of u_{ef}, v_{ef}, and w_{ef} but of their time derivatives in Eqs. (26), (27), and (31) is in accordance with boundary-layer approximation. There are three time scales in unsteady flows, the convection time $T_c=L/U_0$, the diffusion time $T_d=L^2/\nu$, and the imposed time T_i, where L and U_0 are the characteristic length and velocity of the flow, and the ratio $T_d/$

2) To be consistent with (26) and (27), boundary conditions (28) should be understood as

$$u\to u_{ep},\quad w\to w_{ep},\ \text{as}\ y\to\infty,$$

but,

$$\frac{\partial u}{\partial t}\to\frac{\partial u_{ep}}{\partial t}+\frac{\partial u_{ef}}{\partial t},\quad \frac{\partial w}{\partial t}\to\frac{\partial w_{ep}}{\partial t}+\frac{\partial w_{ef}}{\partial t},$$

as $y\to\infty$.

T_c gives the Reynolds number Re and T_c/T_i the Strouhal number St. The unsteady boundary-layer flow is not controlled by the convection time but by the imposed time (say, the elapsed time t itself in a starting flow and the period $1/\omega$ in an oscillating boundary layer) and δ/L becomes of order $1/\sqrt{ReSt}$. Thus, velocities, u_{ef}, v_{ef}, and w_{ef}, are all of order $U_0/\sqrt{ReSt}$ and will be small, but their time-derivatives, $\partial u_{ef}/\partial t$, $\partial v_{ef}/\partial t$, and $\partial w_{ef}/\partial t$, are of order $(U_0\delta/L)/T_i = (U_0^2/L)\sqrt{St/Re}$ and can be large depending on the ratio St/Re. Similarly, $\partial\phi_f/\partial t$ is of order $U_0^2\sqrt{St/Re}$. In other words, the external irrotational velocity field around the unsteady boundary layer will be nearly the same as that of the inviscid flow around the body, but the pressure field can be far from that of the inviscid flow, and neither of terms on the right hand side of Eq. (31) can be neglected from the beginning.

4. Boundary-Layer Formation Around a Circular Cylinder After an Impulsive Start

Suppose that a circular cylinder of radius a starts to move suddenly with constant velocity $-U_0\boldsymbol{k}$ in a direction at right angles to its axis. Then the free-stream velocity u_{ep} becomes

$$u_{ep}=2U_0\sin\frac{x}{a},$$

where x is a distance from the front stagnation point measured along the cylinder surface, and Eqs. (26) and (10) become

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\nu\frac{\partial^2 u}{\partial y^2} =\frac{\partial u_{ef}}{\partial t}+\frac{2U_0^2}{a}\sin\frac{2x}{a} \tag{32}$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{33}$$

respectively.

Introducing boundary-layer variables

$$\tau=\frac{U_0 t}{a},\ \xi=\frac{x}{a},\ \eta=\frac{y}{2\sqrt{\nu t}},$$

$$\tilde{u}=\frac{u}{U_0},\ \tilde{u}_{ep}=\frac{u_{ep}}{U_0},\ \tilde{u}_{ef}=\frac{u_{ef}}{U_0}\frac{a}{2\sqrt{\nu t}},$$

$$\tilde{v}=\frac{v}{U_0}\frac{a}{2\sqrt{\nu t}},\ \tilde{v}_{ef}=\frac{v_{ef}}{U_0}\frac{a}{2\sqrt{\nu t}},$$

$$\tilde{\phi}_f=\frac{\phi_f}{2U_0\sqrt{\nu t}},$$

Eqs. (32) and (33) are transformed into

$$\tau\frac{\partial\tilde{u}}{\partial\tau}-\frac{\eta}{2}\frac{\partial\tilde{u}}{\partial\eta}-\frac{1}{4}\frac{\partial^2\tilde{u}}{\partial\eta^2}=\tau\Big(\frac{1}{\sqrt{Re\tau}}\tilde{u}_{ef}+\frac{2\sqrt{\tau}}{\sqrt{Re}}\frac{\partial\tilde{u}_{ef}}{\partial\tau}-\tilde{u}\frac{\partial\tilde{u}}{\partial\xi}-\tilde{v}\frac{\partial\tilde{u}}{\partial\eta}+2\sin 2\xi\Big) \tag{34}$$

and

$$\frac{\partial\tilde{u}}{\partial\xi}+\frac{\partial\tilde{v}}{\partial\eta}=0, \tag{35}$$

where the Reynolds number Re is based on the oncoming velocity U_0 and the radius a of the cylinder. The equation of continuity (35) can be integrated by introducing a stream function $\tilde{\psi}$ such that

$$\tilde{u}=\frac{\partial\tilde{\psi}}{\partial\eta}\ \text{and}\ \tilde{v}=-\frac{\partial\tilde{\psi}}{\partial\xi},$$

and Eq. (34) becomes

$$\tau\frac{\partial^2\tilde{\psi}}{\partial\eta\partial\tau}-\frac{\eta}{2}\frac{\partial^2\tilde{\psi}}{\partial\eta^2}-\frac{1}{4}\frac{\partial^3\tilde{\psi}}{\partial\eta^3}=\tau\Big(\frac{1}{\sqrt{Re\tau}}\tilde{u}_{ef}+2\frac{\sqrt{\tau}}{\sqrt{Re}}\frac{\partial\tilde{u}_{ef}}{\partial\tau}-\frac{\partial\tilde{\psi}}{\partial\eta}\frac{\partial^2\tilde{\psi}}{\partial\eta\partial\xi}+\frac{\partial\tilde{\psi}}{\partial\xi}\frac{\partial^2\tilde{\psi}}{\partial\eta^2}+2\sin 2\xi\Big) \tag{36}$$

Boundary conditions for $\tilde{\psi}$ become,[3] from (28) and (29),

$$\frac{\partial\tilde{\psi}}{\partial\eta}\to\frac{2\sqrt{\tau}}{\sqrt{Re}}\tilde{u}_{ef}+2\sin\xi,\ \text{as}\ \eta\to\infty, \tag{37}$$

and

$$\frac{\partial\tilde{\psi}}{\partial\eta}=0,\ \frac{\partial\tilde{\psi}}{\partial\xi}=0,\ \text{when}\ \eta=0 \tag{38}$$

The external stream velocity $\tilde{u}_{ef}$ can be computed from

$$-\tilde{u}_{ef}=-\frac{1}{r}\frac{\partial\tilde{\phi}_f}{\partial\theta}, \tag{39}$$

3) Eq. (36) (and also (26) and (27) together with other equations derived from these) is accurate only to the order of boundary-layer approximation. Thus, when the condition (37) is inserted into (36), there appears the discrepancy of order $\tau^{3/2}/\sqrt{Re}$. See also footnote 2.

and the velocity potential $\tilde{\phi}_f$ from the Laplace equation in plane polar coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{\phi}_f}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\tilde{\phi}_f}{\partial\theta^2}=0, \qquad (40)$$

where the modulus r is non-dimensionalized by the radius a and the azimuthal angle θ is from the $\boldsymbol{k}$-axis. The boundary conditions (21) and (22) also become

$$-\nabla\tilde{\phi}_f\Big|_{r\to\infty}=0 \text{ and } -\frac{\partial\tilde{\phi}_f}{\partial r}\Big|_{r=1}=\tilde{v}_{ef}, \qquad (41)$$

where, from (30),

$$\tilde{v}_{ef}=\lim_{\eta\to\infty}\left[\tilde{v}+2\eta\left\{\left(\frac{\tau}{Re}\right)^{1/2}\frac{\partial\tilde{u}_{ef}}{\partial\xi}+\cos\xi\right\}\right] \qquad (42)$$

Equations and boundary conditions, (36) to (42), give sufficient equations and boundary conditions for $\tilde{\psi}$, $\tilde{\phi}_f$, $\tilde{u}_{ef}$, and $\tilde{v}_{ef}$.

Blasius(1908) assumed that the solution for the unsteady boundary layer can be expanded in powers of the time. However, results we have got up to now suggest that the solution should be expanded in powers of $\sqrt{\tau}$ rather than τ. Thus let put

$$\tilde{\psi}=\sum_{n=0}^{\infty}\psi_n(\xi,\eta)\tau^{n/2}, \qquad (43)$$

$$\tilde{\phi}_f=\sum_{n=0}^{\infty}\phi_n(r,\theta)\tau^{n/2}, \qquad (44)$$

$$\tilde{u}_{ef}=\sum_{n=0}^{\infty}u_n(\xi)\tau^{n/2}, \qquad (45)$$

and

$$\tilde{v}_{ef}=\sum_{n=0}^{\infty}v_n(\xi)\tau^{n/2} \qquad (46)$$

Inserting Eqs. (43) and (45) into Eq. (36) and collecting terms with the same power of τ, we have equations for ψ_n's

$$\frac{\eta}{2}\frac{\partial^2\psi_0}{\partial\eta^2}+\frac{1}{4}\frac{\partial^3\psi_0}{\partial\eta^3}=0, \qquad (47)$$

$$\frac{1}{2}\frac{\partial\psi_1}{\partial\eta}-\frac{\eta}{2}\frac{\partial^2\psi_1}{\partial\eta^2}-\frac{1}{4}\frac{\partial^3\psi_1}{\partial\eta^3}=\frac{1}{\sqrt{Re}}u_0, \qquad (48)$$

$$\frac{\partial\psi_2}{\partial\eta}-\frac{\eta}{2}\frac{\partial^2\psi_2}{\partial\eta^2}-\frac{1}{4}\frac{\partial^3\psi_2}{\partial\eta^3}=\frac{2}{\sqrt{Re}}u_1$$
$$-\frac{\partial\psi_0}{\partial\eta}\frac{\partial^2\psi_0}{\partial\eta\partial\xi}+\frac{\partial\psi_0}{\partial\xi}\frac{\partial^2\psi_0}{\partial\eta^2}+2\sin2\xi, \qquad (49)$$

$$\vdots$$

Boundary conditions for ψ_n's are

$$\psi_n=\frac{\partial\psi_n}{\partial\eta}=0 \text{ at } \eta=0, \qquad (50)$$

and

$$\begin{cases}\dfrac{\partial\psi_0}{\partial\eta}\to 2\sin\xi, \\ \dfrac{\partial\psi_n}{\partial\eta}\to\dfrac{2u_{n-1}}{\sqrt{Re}}(n\neq0),\end{cases} \text{ as } \eta\to\infty \qquad (51)$$

The external stream velocity u_n can be calculated from the solution of the external irrotational flow, i. e.,

$$-u_n=-\frac{1}{r}\frac{\partial\phi_n}{\partial\theta}\Big|_{r=1},$$

where the potential ϕ_n of the external irrotational flow satisfies the Laplace equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi_n}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\phi_n}{\partial\theta^2}=0,$$

and boundary conditions

$$-\nabla\phi_n|_{r\to\infty}=0, \text{ and } -\frac{\partial\phi_n}{\partial r}\Big|_{r=1}=v_n$$

From (42) we have also

$$v_0=\lim_{\eta\to\infty}\left[-\frac{\partial\psi_0}{\partial\xi}+2\eta\cos\xi\right]$$

and

$$v_n=\lim_{\eta\to\infty}\left[-\frac{\partial\psi_n}{\partial\xi}+\frac{2\eta}{\sqrt{Re}}\frac{du_{n-1}}{d\xi}\right]$$
$$(n=1,2,3,\cdots)$$

This complicated system of equations can be solved succesively in the order

$$\psi_0\to v_0\to\phi_0\to u_0\to$$
$$\psi_1\to v_1\to\phi_1\to u_1\to$$
$$\psi_2\to\cdots$$

When the procedure is carried out up to the terms in τ, the solutions become[4)]

$$\tilde{\psi}=2\sin\xi\times\left\{G_0(\eta)+\frac{2G_1(\eta)}{\sqrt{\pi Re}}\sqrt{\tau}+\left[\frac{G_2(\eta)}{Re}+2H_2(\eta)\cos\xi\right]\tau\right\}, \qquad (52)$$

$$\tilde{u}=2\sin\xi\times\left\{G_0'(\eta)+\frac{2G_1'(\eta)}{\sqrt{\pi Re}}\sqrt{\tau}+\left[\frac{G_2'(\eta)}{Re}+2H_2'(\eta)\cos\xi\right]\tau\right\}, \qquad (53)$$

4) As Eq. (36) has already the error of order $\tau^{3/2}/\sqrt{Re}$ (footnote 3), it is meaningless to compute terms of higher orders.

$$\tilde{\phi}_f=-\left(\frac{2}{\sqrt{\pi}}+\frac{\sqrt{\tau}}{\sqrt{Re}}\right)\frac{\cos\theta}{r}, \quad (54)$$

$$\tilde{u}_{ef}=\left(\frac{2}{\sqrt{\pi}}+\frac{\sqrt{\tau}}{\sqrt{Re}}\right)\sin\xi, \quad (55)$$

and

$$\tilde{v}_{ef}=\left(\frac{2}{\sqrt{\pi}}+\frac{\sqrt{\tau}}{\sqrt{Re}}\right)\cos\xi, \quad (56)$$

where

$$G_0(\eta)=-\frac{1}{\sqrt{\pi}}+\eta+\text{ierfc}\,\eta,$$

$$G_0'(\eta)=1-\text{erfc}\,\eta,$$

$$G_1(\eta)=-\frac{\sqrt{\pi}}{4}+\eta+\sqrt{\pi}\ \text{i}^2\text{erfc}\,\eta,$$

$$G_1'(\eta)=1-\sqrt{\pi}\,\text{ierfc}\,\eta,$$

$$G_2(\eta)=-\frac{2}{3\sqrt{\pi}}+\eta+4\ \text{i}^3\text{erfc}\,\eta,$$

$$G_2'(\eta)=1-4\ \text{i}^2\text{erfc}\,\eta,$$

$$\begin{aligned}H_2(\eta)=&\frac{1}{\sqrt{\pi}}\left(\frac{4\sqrt{2}}{3}-1-\frac{4}{9\pi}\right)-\frac{1}{2\sqrt{\pi}}e^{-\eta^2}\\&+\frac{2}{3\sqrt{\pi}}\text{erfc}\,\eta-2\ \text{ierfc}\,\eta+\left(6+\frac{8}{3\pi}\right)\text{i}^3\text{erfc}\,\eta\\&+\frac{2}{3}\left(\frac{2}{\sqrt{\pi}}e^{-\eta^2}\text{erfc}\,\eta-\frac{2\sqrt{2}}{\sqrt{\pi}}\ \text{erfc}\sqrt{2}\eta\right.\\&\left.+\text{erfc}\,\eta\ \text{ierfc}\,\eta-\text{ierfc}\,\eta\ \text{i}^2\text{erfc}\ \eta\right),\end{aligned}$$

and

$$\begin{aligned}H_2'(\eta)=&\left(-\frac{4}{3\pi}+\frac{1}{\sqrt{\pi}}\eta\right)e^{-\eta^2}+2\ \text{erfc}\,\eta\\&-\left(6+\frac{8}{3\pi}\right)\text{i}^2\text{erfc}\,\eta\\&+2\left(\text{ierfc}^2\eta-\text{erfc}\,\eta\ \text{i}^2\text{erfc}\,\eta\right)\end{aligned}$$

The repeated integrals of the error function are defined as (Abramowitz and Stegun, 1965)

$$\text{i}^n\text{erfc}\,\eta=\int_\eta^\infty \text{i}^{n-1}\text{erfc}\,\eta d\eta \quad (\eta=0,1,2,\cdots),$$

$$\text{i}^{-1}\text{erfc}\,\eta=\frac{2}{\sqrt{\pi}e^{-\eta^2}},\ \text{and}\ \text{i}^0\text{erfc}\,\eta=\text{erfc}\,\eta$$

It follows from (53) that the frictional stress τ_w at the surface becomes

$$\begin{aligned}C_f\equiv&\frac{\tau_w}{\frac{1}{2}\rho U_0^2}\\=&\frac{4\sin\xi}{\sqrt{\pi}Re}\left\{\left(\frac{\tau}{Re}\right)^{-1/2}+\sqrt{\pi}+2\right.\\&\left.\left[1+\left(1+\frac{4}{3\pi}\right)Re\cos\xi\right]\left(\frac{\tau}{Re}\right)^{1/2}\right\}\end{aligned} \quad (57)$$

The external free-stream velocity varies as

$$\frac{u_e}{U_0}=2\sin\xi\left\{1+\frac{2}{\sqrt{\pi}}\left(\frac{\tau}{Re}\right)^{1/2}+\frac{\tau}{Re}\right\}, \quad (58)$$

and the pressure p_s on the surface as

$$\begin{aligned}C_p\equiv&\frac{p_s-p_\infty}{\frac{1}{2}\rho U_0^2}\\=&-1+2\cos2\xi+\frac{4\cos\xi}{\sqrt{\pi}Re}\left\{\left(\frac{\tau}{Re}\right)^{-1/2}+\sqrt{\pi}\right\}\end{aligned} \quad (59)$$

Putting $\xi=0$ in the above equation we have

$$C_{p0}\equiv\frac{p_s-p_\infty}{\frac{1}{2}\rho U_0^2}=1+\frac{4}{Re}+\frac{4}{\sqrt{\pi Re\tau}}, \quad (60)$$

for the pressure p_0 at the front stagnation point.

Integrating Eqs. (57) and (59), we get the friction drag D_f and form drag D_p per unit length of the cylinder as

$$\begin{aligned}C_{Df}\equiv&\frac{D_f}{\rho U_0^2 a}\\=&\frac{2\sqrt{\pi}}{Re}\left\{\left(\frac{\tau}{Re}\right)^{-1/2}+\sqrt{\pi}+2\left(\frac{\tau}{Re}\right)^{1/2}\right\}\end{aligned} \quad (61)$$

and

$$C_{Dp}\equiv\frac{D_p}{\rho U_0^2 a}=\frac{2\sqrt{\pi}}{Re}\left\{\left(\frac{\tau}{Re}\right)^{-1/2}+\sqrt{\pi}\right\} \quad (62)$$

Thus the total drag D becomes

$$C_D\equiv\frac{D}{\rho U_0^2 a}=\frac{4\sqrt{\pi}}{Re}\left\{\left(\frac{\tau}{Re}\right)^{-1/2}+\sqrt{\pi}\right\} \quad (63)$$

When adding C_{Df} and C_{Dp}, we have dropped the last term with order $O(\tau^{1/2})$ in Eq. (61), as the equation for C_{Dp} is accurate only up to order O (1).

All these solutions from (52) to (63), except those for the pressure (Eqs. (59) and (60)) and form drag (Eqs. (62) and (63)), asymptote to corresponding ones of Blasius, when the Reynolds number Re goes to infinity. Note also that these solutions are held only when both τ and τ/Re are small, i.e.,

$$\tau\ll1\ \text{and}\ \frac{\tau}{Re}\ll1 \quad (64)$$

The first condition is obvious. The second is necessary, since the boundary-layer thickness which is of oredr $\sqrt{\nu t}$ must be small compared to the radius a.

It seems extremely difficult to measure the instant pressure and drag directly just after the start of flow; in most experimental results (say, Schwabe's one (Stuart, 1963, p. 374)) the pressure is estimated from the instant velocity distribution without considering the effect of the unsteady term $\partial v/\partial t$. However, some numerical results agree well with present results. In Fig. 2, pressure distributions on the cylinder surface at low Reynolds numbers are presented together with numerical results by the present author (Koh and Bradshaw, 1993). For comparison we plotted the curves up to a normalized time of 10, although the expression (59) is not valid at this time. The analytic predictions and numerical ones are in fair agreement at small times and both give the asymmetric pressure distribution with strong favourable pressure gradient around the cylinder, though it shows a tendency that, at $Re=10$, the anti-symmetric pressure distribution at $\tau=0$ evolves to that of the inviscid flow, as time goes on. The numerical results, however, show that this trend is only up to $\tau\approx1.0$ and then the pressure approaches to that of the steady flow with standing eddies.

Note that in Fig. 2 the pressure coefficient is normalized by that at the front stagnation point whose magnitude decreases in inverse proportion of $\sqrt{\tau}$ (Fig. 3) at small τ, which means that the form drag decreases also from infinity at $\tau=0$. Time history of the pressure at the front stagnation point and total drag are in Figs. 3 and 4, respectively, with numerical results of Koh and Bradshaw(1993). Analytic results of Bar-Lev and

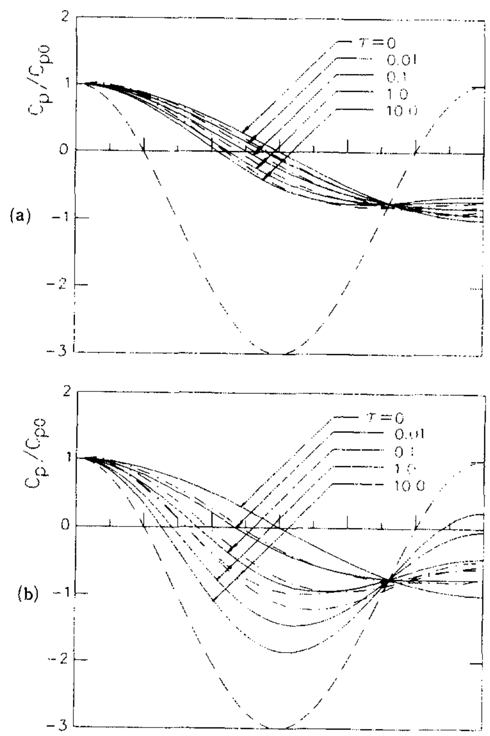

Fig. 2 Tim development of the pressure distribution around a circular cylinder. (a) $Re=1$, (b) $Re=10$. ———, present analysis ; —·—, numerical method (Koh and Bradshaw 1993) ; — — —, inviscid flow

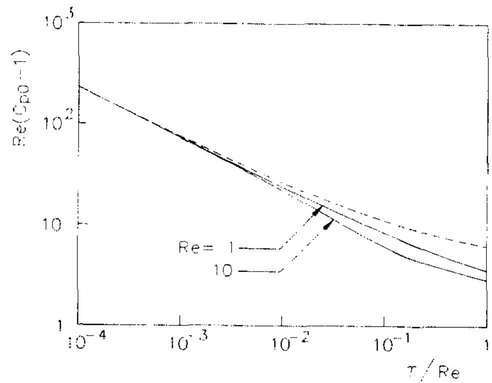

Fig. 3 Pressure coefficient at the front stagnation point of a circular cylinder which is set to move impulsively. — — —, present analysis ; ———, numerical method (Koh and Bradshaw 1993).

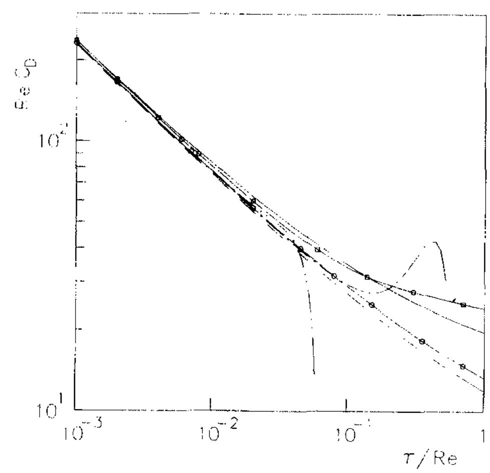

Fig. 4 Drag coefficient of a circular cylinder which is set to move impulsively. ———, present analysis ; —o—, Koh and Bradshaw(1993), $Re=1$; —□—, Koh and Bradshaw(1993), $Re=10$; — — —, Bar-Lev and Yang(1975) ; —·—, Collins and Dennis(1973), $Re=1$; —··—, Collins and Dennis(1973), $Re=10$.

Yang(1975) and Collins and Dennis(1973) are included also in Fig. 4 for comparison. All results are in fair agreement with each other up to $\tau/Re \approx 0.01$, though the present result gives slightly greater values comparing to other results. In the figure, the Reynolds numbers are not elucidated for the present and Bar-Lev and Yang's results ; both give expressions (Eq. (63) above and Eq. (49) of Bar-Lev and Yang(1975)) for ReC_D only up to $O(1)$ terms which do not depend on Re but only on τ/Re. Higher order terms will depend on both Re and τ/Re, but results of Collins and Dennis(1973) show that improvements are minor.

5. Conclusions

Using the momentum theorem, it is shown that, in an unsteady incompressible flow, the friction drag is always accompanied by the form drag whose magnitude is comparable to that of the former and, thus, that the pressure distribution around the unsteady boundary layer can be far from that of the inviscid irrotational flow. The unsteady boundary-layer equations and boundary conditions for the external potential flow are modified accordingly and the modified equations are used to analyse the flow around a circular cylinder which is set impulsively to move in a constant velocity. The results are different from Blasius classical ones in several respects.

(1) The solutions are in power series of $\sqrt{\tau}$ rather than τ itself, where τ is the dimensionless time elapsed since the onset of motion.

(2) There appears a strong favourable pressure gradient throughout the whole cylindrical surface just after the start of motion.

(3) Thus, like the friction drag, the form drag is not zero but infinite at the start and then decreases in inverse proportion to $\sqrt{\tau}$.

(4) And present results except those about the form drag and pressure distribution asymptote to Blasius ones when the Reynolds number goes to infinity.

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